Section 2.5: Autonomous Equations and Population dynamics

Definition: An equation of the form

$$\frac{dy}{dt} = f(y) \qquad (*)$$

is called autonomous. In other words, the independent variable does not appear explicitly.

Example: ~~A spec~~ ~~The first order equation~~
The special type of first order linear equation w/ constant coefficients:

$$\frac{dy}{dt} = ay + b \text{ is autonomous.}$$

The main purpose of this section is ~~give some~~ show how geometrical methods can be used to obtain important qualitative information directly from the equation $(*)$, without solving them.

The simplest ~~growth~~ model describing the population is the exponential Growth model

$$\frac{dy}{dt} = ry,$$

where r is called rate of growth or decline. is a constant (depending on the sign).

We can easily see that the solution is ~~牢~~ given by

$$y = y_0 \cdot e^{rt}$$

Logistic Growth: We need to take account of the fact that the growth rate actually depends on the population, instead of a constant r. Thus we get the following equation:

$$\frac{dy}{dt} = h(y) \cdot y.$$

~~We ne~~ The function $h(y)$ should satisfy the following conditions

(1). $h(y) \approx r > 0$ when y is small.

(2). $h(y)$ decrease as y grows large (limitation of food, water, ...)

(3). $h(y) < 0$ when y is sufficiently large.

The simplest function that satisfies these properties is

$$h(y) = r - ay \qquad \cancel{(**)}$$

Then we obtain the equation

$$\frac{dy}{dt} = (r - ay)y$$

$$= r\left(1 - \frac{y}{K}\right) \cdot y \qquad (**). \qquad \left(K = \frac{r}{a}\right).$$

This equation is called the logistic equation.

The constant r is called the intrinsic growth rate. That is the growth rate in the absence of any limiting factors.

Before solving this equation, ~~we~~ first ~~ske~~ draw a sketch of the solutions.

The simplest type ~~of~~ of solution is constant functions:

$$\frac{dy}{dt} = r\left(1-\frac{y}{K}\right)y$$

$$\Longrightarrow \quad \frac{dy}{dt} = 0 \quad \Longrightarrow \quad y = 0 \text{ or } K.$$

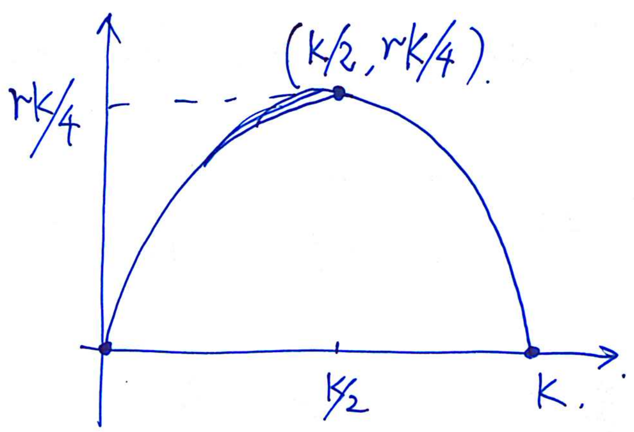
These constant solutions are called equilibrium solutions.

More generally, the equilibrium ~~of~~ solutions ~~of~~ of the autonomous equations

$$\frac{dy}{dt} = f(y)$$

are the same as the ~~solu~~ ~~constant~~ ~~solutions~~ zeroes. of $f(y) = 0$.

To visualize other solutions of the logistic equation, we first draw the graph of $f(y)$:

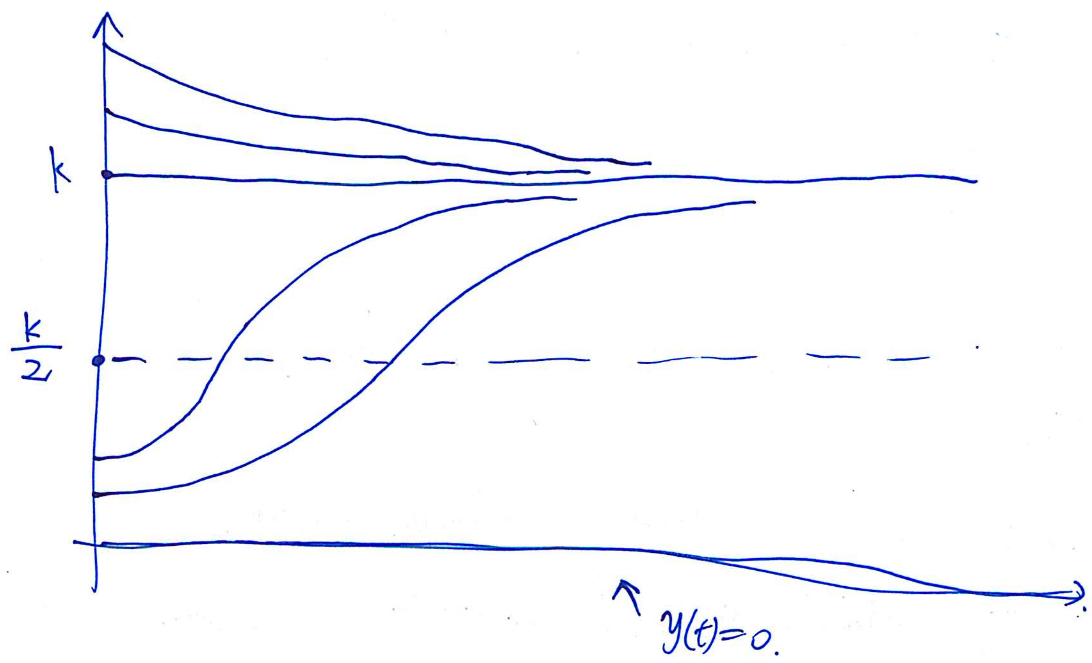


The graph is a parabola: the graph shows

1). $\frac{dy}{dt} > 0$ for $0 < y < K$, then $\frac{dy}{dt} > 0$, y is increasing

2). $y > K$: $\frac{dy}{dt} < 0$ y is decreasing.

Let's now sketch the solutions (no given initial condition).



We can apply Theorem 2.4.2 about uniqueness of solutions to get some information. None of the curves of solutions can intersect each other, since $f(y) = r(1-\frac{y}{k})y$ is continuous, and $\frac{\partial f}{\partial y}$ is also continuous.

The dashed line indicates the place where the concavity changes, or in other words the inflection points. Recall that inflection points are given by ~~solving~~ finding

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\frac{dy}{dt} = \frac{df(y)}{dt} = f'(y)\cdot\frac{dy}{dt} = f'(y)f(y).$$

We know that: The graph of y is

concave up if $y''>0$.

concave down if $y''<0$.

Inflection points: where the concavity changes. or (when y'' changes sign)

In our case: the solution curves are

1). concave up for $0<y<K/2$, (where f, f' are both positive).

2). concave down for $K/2<y<K$, ($f>0$, $f'<0$).

3). concave up for $y>K$ (f, f' both negative).

Remark: The constant K is called the <u>environmental carrying capacity</u>.

Let's now try to solve the logistic equation:

suppose $y \neq 0$, $y \neq K$:

$$\frac{dy}{(1-y/K)\cdot y} = r\,dt.$$

The LHS is

$$\left(\frac{1}{y} + \frac{1/K}{1-y/K}\right) dy = r\,dt$$

By integrating both sides, we obtain

$$\ln|y| - \ln\left|1-\frac{y}{K}\right| = rt + C.$$

Taking exponential of both sides:

$$\frac{y}{1-y/K} = C\cdot e^{rt}$$

With an initial condition $y(0)=y_0$, we obtain

$$\frac{y_0}{1-y_0/k} = C$$

$$\Rightarrow y(t) = \frac{y_0 K}{y_0 + (K - y_0) \cdot e^{-rt}}$$

We can see that

$$\lim_{t\to\infty} y(t) = \frac{y_0 K}{y_0} = K. \text{ (if } y_0 \neq 0)$$

Conclusion: 1. for each $y_0 > 0$, the solution approaches the equilibrium solution $y \equiv K$, so $y \equiv k$ is called an asymptotically stable solution.

2. $y \equiv 0$ is called an unstable equilibrium solution.

Section 2.6 : Exact equations and integrating factors.

2.6

In this section, we consider a class of equations known as exact equations for which there is also a well-defined method to solve.

Example: Solve the differential equation

$$2x + y^2 + 2xyy' = 0. \quad (*)$$

Note that this equation is not linear (y^2 term), non-separable.

Observation: Let $\Psi(x,y) = x^2 + xy^2$.

$$\Longrightarrow \frac{\partial \Psi}{\partial x} = 2x + y^2, \quad 2xy = \frac{\partial \Psi}{\partial y}$$

And it follows the original equation (*) can be written as

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx} = 0.$$

Now y is a function of x, then $\Psi(x,y) = \Psi(x, y(x))$ is also a function of x.

$$\frac{d\Psi}{dx} = \frac{d}{dx}\left(x^2 + xy^2\right) = 0.$$

$$\Longrightarrow \Psi(x,y) = x^2 + xy^2 = C,$$ where C is an arbitrary constant.

Recall that the key step is ~~to~~ observe the ~~Ψ~~ there is a function.

~~Ψ~~ ~~that~~ More generally, let the differential equation

$$M(x,y) + N(x,y)y' = 0 \text{ be given.}$$

Suppose that we can identify a function Ψ s.t.

$$\frac{\partial \Psi}{\partial x}(x,y) = M(x,y), \quad \frac{\partial \Psi}{\partial y} = N(x,y),$$

then $\Psi(x,y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of x.

In this case, the equation

$M(x,y) + N(x,y)y' = 0$ is called an exact differential equation.

Question: (1). How to ~~find~~ determine if an ~~function~~ equation is exact?

(2). How to find the ~~|~~ corresponding function $\Psi(x,y)$?

Theorem 2.6.1: Let the functions M, N, M_y, N_y (whose subscripts ~~denote~~ denote partial derivaties $M_y = \frac{\partial M}{\partial y}$) be continuous in a retagular region $\alpha < x < \beta$, $\gamma < y < \delta$, then the equation ~~$M(x,y)+N(x,y)$~~

$$M(x,y) + N(x,y) \cdot y' = 0.$$

is an exact differential equation in R ~~if~~ and only if

$$M_y(x,y) = N_x(x,y).$$ at each point in R.

Proof: The proof consists of two parts:

Part 1: Suppose the equation is exact, meaning that $\exists$ a function $\Psi(x,y)$, s.t. $M = \Psi_x$, $N = \Psi_y$.

$\Rightarrow$ This implies $M_y(x,y) = \Psi_{xy}(x,y)$ and $N_x(x,y) = \Psi_{yx}(x,y)$.

Since M_y and N_x are continuous $\Rightarrow$ $M_y = N_x$ (order of derivative ~~change~~ does not matter);

Part 2: Suppose $M_y(x,y) = N_x(x,y)$, we want to find a function $\Psi(x,y)$, s.t. $\Psi_x = M$, $\Psi_y = N$.

The proof also gives the method of computation.

To find $\Psi(x,y)$, We begin by integrating the equation

$$\frac{\partial \Psi}{\partial x}(x,y) = M(x,y)$$ w.r.t x, holding y as a constant.

We thus obtain $\Psi(x,y) = Q(x,y) + h(y)$.

$\neq f$

$$= \int_{x_0}^{x} M(s,y)\,ds + h(y).$$

Now we need to show that we can always choose a function $h(y)$, s.t. the equation ~~the~~ $\frac{\partial \Psi}{\partial y}(x,y) = N(x,y)$:

Now that ~~$Q(x,y)$~~ $\Psi(x,y) = Q(x,y) + h(y)$.

$$\Longrightarrow \frac{\partial \Psi}{\partial y}(x,y) = \frac{\partial Q}{\partial y} + h'(y) = N(x,y)$$

$$\Longrightarrow \boxed{h'(y) = N(x,y) - \frac{\partial Q}{\partial y}(x,y)}. \quad (**)$$

In particular, the RHS must be independent of x, let's verify this by differentiating it with respect to x:

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial x}\frac{\partial Q}{\partial y}(x,y)$$

$$= \frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial y}\frac{\partial Q}{\partial x}(x,y)$$

$$\overset{\text{using } \frac{\partial Q}{\partial x} = M}{=\!=\!=} \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

So the equation $(**)$ saying

$$h'(y) = \text{a function of } y \text{ only}.$$

$\Longrightarrow$ We obtain $h(y)$ by integration.

Example: Solve the differential equation

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$$

$$M(x,y) = y\cos x + 2xe^y, \qquad M_y(x,y) = \cos x + 2xe^y$$

$$N(x,y) = \sin x + x^2e^y - 1. \qquad N_x(x,y) = \cos x + 2xe^y.$$

$$\Longrightarrow M_y = N_x.$$

We need to find $\Psi(x,y)$, s.t. $\Psi_x = M$, $\Psi_y = N$.

$$\Longrightarrow \Psi(x,y) = y\sin x + x^2e^y + h(y).$$

Now the equation $\Psi_y = N$ gives.

$$\sin x + x^2e^y + h'(y) = \sin x + x^2e^y - 1.$$

$$\Longrightarrow h'(y) = -1$$

We let $h(y) = -y$.

$$\Psi(x,y) = y\sin x + x^2e^y - y$$

Hence the solutions are given by

$$y\sin x + x^2e^y - y = C.$$

Integrating factor: Let's turn the equation $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

to the differential form $M(x,y)\,dx + N(x,y)\,dy = 0$.

It might be the case that the equation is not exact. The idea is to multiply some function $\mu(x,y)$ to make

$$\mu M\,dx + \mu N\,dy = 0 \text{ exact.}$$

By the previous theorem, the exactness is ~~equivalent~~ precisely

$$(\mu M)_y = (\mu \cdot N)_x.$$

$$\Longrightarrow M\mu_y \cancel{+ \mu} - N\mu_x + (M_y - N_x)\mu = 0.$$

This ~~will~~ is a PDE.

We can only solve this problem in very special cases:

For instance, assume that μ is only a function of x only.

$$\Longrightarrow \cancel{\mu_x}\ \mu_x = \frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu.$$

(So we assume that $\frac{M_y - N_x}{N}$ is ~~only~~ a function of x). And we can find μ.

Example: Solve the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

$$\Rightarrow (3xy + y^2)\,dx + (x^2 + xy)\,dy = 0.$$

$M = 3xy + y^2$,

$N = x^2 + xy$,

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{x+y}{x(x+y)} = \frac{1}{x}.$$

$$\Rightarrow \frac{d\mu}{dx} = \frac{\mu}{x} \Rightarrow \mu(x) = x.$$

Then the equation $x(3xy + y^2) + x(x^2 + xy)y' = 0$ will be exact.

~~Remark:~~ (first) Remark: From the equation $\frac{d\mu}{dx} = \frac{M_y - N_x}{N} \cdot \mu$

we can see that suppose $M_y = N_x$, no integrating factor will be necessary. $\frac{d\mu}{dx} = 0 \Rightarrow$ we can take $\mu = 1$.